Hohol, M., Bażela, N. (2024). Mathematical cognition. In J. Bremer (Ed.), *Cognitive science* (pp. 223–241). Ignatianum University Press. https://doi.org/10.35765/slowniki.436en

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https://doi.org/10.35765/slowniki.436en

Mathematical cognition

Summary

DEFINITION OF THE TERM: Mathematical cognition is an area of science that deals with mathematical activity in the mind (such as processing numbers and geometry), and the biological basis for human mathematical competence. Mathematical cognition is an area of interdisciplinary research founded on psychology and neuroscience.

HISTORICAL ANALYSIS OF THE TERM: The study of mathematical cognition follows a tradition of philosophical inquiry concerning mathematical intuition. While philosophers like Plato, Descartes, Kant, and Poincaré viewed this intuition as innate, Helmholtz thought it is shaped by experience. In his pioneering research on the development of mathematical cognition in children, Piaget continued to use the category of intuition and pointed out that knowledge of numbers and space is not built through observation but in action.

DISCUSSION OF THE TERM: Mathematical cognition is based on systems of core knowledge that are evolutionarily old, develop early in the course of individual development, and are culturally universal. These systems enable people to accurately assess the size of small sets of objects, estimate the size of larger sets, process shapes, and navigate spatially from an early age. However, these systems have limitations which are overcome – with the support of language – when learning numerals and spatial vocabulary. The research findings further indicate that the habit of finger-counting plays an important role in the transition from core knowledge systems to symbolic mathematics.

SYSTEMATIC REFLECTION WITH CONCLUSIONS AND RECOMMENDATIONS: The study of mathematical cognition provides knowledge about one of the main research areas in the social sciences: the interactions between nature and culture. It also has implications for education, where it aids the creation of guidelines for supporting students with learning difficulties.

Keywords: geometry, numbers, spatial-numerical associations, innateness, core knowledge

Definition of the term

From the perspective of cognitive science, ‘mathematical cognition’ can be understood in two ways: as the *explanans* and as the *explanandum* of cognitive processes (Hohol, 2020). In the former, the mind is explained with the aid of mathematics, particularly modelling various cognitive processes using mathematical tools. In the latter, cognitive activity associated with the use of mathematics is explained by enlisting the methods and theories used in empirically oriented cognitive science. This approach involves interdisciplinary research grounded in experimental psychology and cognitive neuroscience, complemented by computational modelling, cognitive anthropology, cognitive ethology, and educational research.

In this article, we focus primarily on ‘mathematical cognition’ understood as *explanandum*, using the approach proposed by Rafael Núñez and George Lakoff (2005). Within this approach, the subject matter of studies on mathematical cognition includes

> some aspect of the psychological, neurological, or educational reality involved in some mathematical behaviour, performance, or competence of a person. The subject matter is defined at the level of an individual or at the level of an individual’s nervous system. Mathematics per se is untouched (Núñez & Lakoff, 2005, p. 110).

The scope covers “what is usually meant by ‘mathematics’, [i.e.,] in general, simple arithmetic, number processing, or numerical calculation”, as well as spatial and quantitative aspects of the environment, the modes of justification and proof practices used in mathematics, and the processing of mathematical objects and structures taught at the academic level. The scope of research on mathematical cognition is broad and includes studies on different populations, from infants to professional mathematicians. The research methods “are mainly standard empirical methods used in behavioral studies in psychology, studies with neuropsychological syndromes, and computational models of numerical processing” (Núñez & Lakoff, 2005, p. 110). Typical questions asked by researchers of mathematical cognition include: How does the mind represent numbers? Is there an innate basis for human numerical and geometrical competence, and if so, what is it?

Which brain structures are involved in the elementary cognitive processing of numbers and geometric properties? Is the cognitive processing of mathematical structures acquired at the academic level, built on elementary numerical-geometric cognition?

Historical analysis of the term

One of the earliest and most seminal works that has shaped our understanding of mathematical theories is Euclid's *Elements* (4^{th}-3^{rd} century BC), in which he presented patterns of logical reasoning and mathematical deduction. He summarised the mathematical knowledge accumulated over centuries by Greek mathematicians who drew on Egyptian and Babylonian achievements. Euclid's systematisation included plane geometry, stereometry, geometric algebra, and number theory, the latter also being geometric in nature. Numbers were treated as segments, while multiplication was seen as a measurement. Hence, geometry remained synonymous with mathematics for centuries, and geometric proofs were regarded a model of rigorous thinking. A frequently occurring term in *The Elements* is 'construction', which is clearly evident in Book I, where the first theorem concerns the construction of an equilateral triangle on a given finite straight line. Although the term 'construction' suggests that mathematical objects do not exist before reasoning is conducted, Euclid's work does not contain philosophical declarations of this kind.

Plato (4^{th} century BC) believed that it was possible, through many years of training, to acquire mathematical knowledge thanks to an innate cognitive ability to grasp elementary knowledge about points, segments, angles, and relations between figures. This ability, traditionally called 'mathematical intuition', can be considered a historical predecessor to 'mathematical cognition' (Hohol, 2020). Intuition, also called 'insight', is linked to the metaphor of 'thinking is seeing', which dates back at least to Plato. Plato's thesis of the intuitiveness of elementary mathematics is illustrated in, among others, his dialogue *Meno*, in which an uneducated slave, guided by questions posed by Socrates, is capable of deductive reasoning, which leads him to the conclusion that a square built on the diagonal of a given square is twice as large (see Hohol, 2020). According to Plato, constructing a geometric figure creates a representation of a mathematical object that is 'seen' by the mind but exists independently of it (see Hohol, 2020; Murawski, 2013). This thesis was defended by Plato's followers, such as Speusipppus of Athens (4th century BC). However, not all Hellenistic philosophers were Platonists. For instance, Aristotle defended the thesis of the empirical nature of mathematical knowledge that arises from the abstraction of mathematical objects from physical objects, while Menaechmus (4th century BC), a mathematician associated with the Platonic Academy, argued that geometrical constructions should be interpreted literally as operations that result in the creation of mathematical objects.

Mathematical intuition became one of the classical themes of modern philosophy. Descartes (1596–1650) claimed that mathematical theorems concerning eternal truths are discovered through intuition, which he called the 'natural light'. Significantly, according to him, the intuition that makes it possible to recognise the clarity and distinctness of mathematical theorems is innate. It is completely independent of the testimony of the senses and can be expressed by the metaphor 'thinking is seeing' (see Hohol, 2020, Murawski, 2013). Discussions on intuition continued in subsequent eras, most notably in the conceptions of Immanuel Kant (1724–1804), who believed that all mathematical theorems have the status of synthetic a priori judgements, which extend our knowledge (in contrast to tautologies) and are completely independent of empirical justification as their truth is recognised intuitively (see Hohol, 2020; Murawski, 2013). Synthetic a priori judgements are possible by linking mathematical constructs to space and time, understood as forms of pure intuition, inherent in our faculty of mind, which filter sensory impressions and organise our cognition. The internal representation of space enables the subject to construct geometry, while the representation of time underpins his ability to construct numbers, which are understood as units attached to one another sequentially in time (see Hohol, 2020).

Henri Poincaré (1854–1912) directly referred to Descartes' and Kant's conceptions of mathematical intuition and substantially modified its understanding (see Hohol, 2020). Poincaré called mathematical intuition the 'innate creativity' of the human mind which allows man to recognise fundamental relations and mathematical theorems as true. Noticing that mathematical cognition involves both conscious and unconscious processing was Poincaré's innovative contribution to the understanding of this concept. He believed that many mathematical theorems arise at the unconscious level but must be supplemented by consciously controlled reasoning. Contrary to Kant, he claimed that the axioms that serve as the starting point of geometrical proofs are not a priori judgements but are adopted by mathematicians by virtue of convention, convenience, utility, and empirical facts. This led Poincaré to recognise that the geometry defined by Euclid's axioms was not the only valid version of geometry, although he acknowledged it as the most convenient. Innate mathematical intuition thus guarantees that reasoning remains correct, and the final results depend on the assumptions made by convention (see Hohol, 2020).

Hermann von Helmholtz (1821–1894) went even further in modifying the previous understanding of mathematical intuition by denying its innate nature and leaning towards radical empiricism. Although he agreed with Kant that sensual experiences and impressions are filtered through the mind, he rejected the transcendentalist viewpoint entirely. According to Helmholtz, the cognitive ability traditionally called mathematical intuition is shaped by accumulating experiences and reinforcing the associated memory traces. This means that the shaping of mathematical intuition can, and even should, be 'broken down' into simpler components; then, it should be shown how mathematical thinking is shaped during individual development. Helmholtz's theses on the empirical origins of mathematical intuition were still philosophical in nature but were inspired by his psychological research (it is worth mentioning here that he was one of the pioneers of experimental psychology), which laid the foundations for empirical research on learning mathematics (see Hohol, 2020).

The term 'mathematical intuition' did not fall out of use with the rise of experimental psychology, although its meaning underwent another significant change. Jean Piaget (1896–1980) played an important role in developing modern empirical research on mathematical cognition. He argued that cognitive development proceeds according to ordered stages, beginning with the sensorimotor stage (from birth to two years), when children begin to understand object permanence. The next stage (ages two to around six), the preoperational stage, is characterised by the inability to take a viewpoint other than one's own. Then (ages seven to twelve), children go through the concrete operational stage, during which they master the basics of logic and become able to take another's viewpoint, which enables them to understand cause-and-effect relationships. Finally, at age twelve, children enter the formal operational stage, which is characterised by the ability to think abstractly.

Piaget devoted other works to the genesis of numerical and geometrical competence, based on a constructivist and action-oriented approach to all mathematical concepts (see Semadeni, 2023). In his opinion, mathematical concepts are not innate and do not emerge suddenly. They are gradually constructed in a child's mind and are based on experience. They are shaped not merely by perceiving the physical world, but by exploring the environment and actively manipulating objects. Piaget argued that arithmetical competence develops in parallel with logical abilities. The concept of number arises as a result of abstraction (i.e., disregarding the physical characteristics of objects, such as colour, size, or shape) based on actions performed on objects, such as the assignment of objects to sets (see Szczygieł, 2017). The development of the concept of number requires internalising the principle of permanence and, in particular, understanding that changing the position of objects in sets does not change their number. In a typical study conducted by Piaget, a child was shown two rows of the same number of objects (e.g., balls), located at the same distance; then, the objects in one of the rows were moved apart, which meant a change in their spatial arrangement but not in their number. Piaget noticed that even five-year-old children incorrectly claimed that pulling objects apart led to an increase in the number of objects. This error does not subside until the age of six or seven, when it marks the complete formation of the concept of number. However, later studies demonstrated that tasks that test the understanding of the principle of object permanence are correctly solved by children at a younger age. This discrepancy was explained by, among other things, the unnaturalness of Piaget's testing conditions (see Szczygieł, 2017). Moreover, using other research paradigms made it possible to study the presence of number representations even in newborns (see the *Discussion of the term* section).

Piaget was convinced that geometrical concepts do not simply derive from observing the world. For example, 'equality' derives from internalising the action of aligning two objects; a 'straight line' has at its root the internalisation of the action of following something, e.g., with one's eyes, without changing direction; an 'angle' derives from the internalisation of two intersecting movements of, e.g., hands (see Hohol, 2020; Semadeni, 2023). Regarding the cognitive origins of geometry, Piaget – as was already mentioned above – did not shy away from the classical notion of intuition. As he wrote in a book he co-authored with Bärbel Inhelder:

> The 'intuition' of space is not a 'reading' or apprehension of the properties of objects, but from the very beginning, an action performed on them. It is precisely because it enriches and develops physical reality instead of merely extracting from it a set of ready-made structures, that action is eventually able to transcend physical limitations and create operational schemata which can be formalised and made to function in a purely abstract, deductive fashion. From the rudimentary sensori-motor activity right up to abstract operations, the development of geometrical intuition is that of an activity, in the fullest sense (Piaget & Inhelder, 1967, p. 449).

Piaget's concept, although largely undermined by later research (see Spelke, 2022; Szczygieł, 2017), was original and historically important as it represented the first attempt to define the stages of a child's arrival at the concept of Euclidean space (see Hohol, 2020). According to Piaget, individual development occurs in reverse order to that of the historical development of geometry as a branch of mathematics: while Euclidean geometry was established in antiquity, projective geometry developed in the 19th century, and topology emerged at the turn of the 20th century, Piaget argued that the most primordial concept of space that is formed in the mind of a child at around the age of three is 'topological space' (although it must be emphasised here that what Piaget meant by topology is far from its strictly mathematical definition (see Hohol, 2020; Semadeni, 2023)). He formulated his conclusions based on experiments in which children distinguished shapes, e.g., by touch, and did well when one of the objects was closed and the other was not. However, the youngest children did not manage to distinguish between a square and a triangle or even between rectilinear and curvilinear figures. On this basis, Piaget concluded that three-year-old children did not yet have the concepts of lengths or angles. While certain Euclidean concepts seem to form at around six or seven years of age, children still see all geometrical relations relative to their viewpoint (which corresponds to the still egocentric preoperational stage of general cognitive development). On this basis, Piaget concluded that the child at this age has a concept of projective space. Finally, at around the age of twelve (which coincides with entry into the formal operational stage), children develop the concept of Euclidean space, which does not refer to a particular arrangement of objects that occupy perceived positions but rather to the organisation of space itself as an invisible 'container' independent of the mobile objects contained within it in which the relations between these objects are independent of the current viewpoint (Piaget & Inhelder, 1967). As is the case with the development of the concept of number, today the thesis of the primacy of topology is contested. For example, Piaget's observation that children first distinguish between open and closed figures is explained by their greater familiarity with certain objects rather than by the most basic nature of topology (see Hohol, 2020). What seems to be still valid, though, is Piaget's thesis that mathematical competence does not derive from mere visual perception of the world but from movement and action in the world.

Discussion of the term

The interdisciplinary scientific study of cognition, today called cognitive science, emerged at the turn of 1960s, and initially covered primarily cognitive psychology, linguistics, and computer science, which were gradually joined by neuroscience, anthropology, and philosophy. The pioneers of cognitive science, such as Allen Newell and Herbert Simon, were vitally interested in mathematical and logical reasoning, which is reflected in what were considered the first artificial intelligence programs: The Logic Theorist and the General Problem Solver, which were capable, among other things, of successfully proving the theorems from Bertrand Russell and Alfred North Whitehead's famous mathematical treatise *Principia mathematica*. Other programs from this period successfully proved theorems of Euclidean geometry. As critics observed, computer programs could achieve analogous mathematical results in a completely different way from humans. To eliminate this problem, designers and programmers gradually began to use psychological data (originally verbal protocols, later reaction time data). Hence, geometric programs started to simulate the interactions that people have with external representations e.g., diagrams (see Hohol, 2020).

However, proving theorems, which is a complex activity involving multiple cognitive processes, did not become the main focus of research on mathematical cognition. Instead, researchers focused on more elementary issues, such as the way the mind represents numbers, which, as it turned out, has much in common with the analogue processing of physical quantities in general. The results of numerous studies demonstrate that, when comparing two perceived physical objects, both humans and animals perform better (i.e., they achieve higher correctness and faster reaction times) when the objects differ significantly in size than when the difference in size is not as large. When subjects compare pairs of objects with a fixed size difference, they find it easier to compare pairs of small objects than pairs of larger objects (see Butterworth, 2022). The former is called the distance effect and the latter the size effect. Importantly, these effects occur when different physical quantities are processed, including the length of a segment or the brightness of an object.

In 1967, Moyer and Landauer described a similar distance effect for numbers (see Brożek & Hohol, 2017; Dehaene, 2011). When people are asked to choose the larger of two numbers (represented by Arabic symbols), they respond faster and make fewer errors when the numbers being compared are separated by a greater distance (e.g., 5 and 9) than when the distance is smaller (e.g., 4 and 5). A size effect for numbers has also been discovered: when the distance between the numbers being compared is constant (e.g., the difference between 3 and 5 and 5 and 7 is 2), people respond faster and make fewer errors when the numbers are smaller. Based on the discovery of these two effects, the concept of the mental number line has been proposed. The distance effect indicates that, at the level of automatic and unconscious cognitive processing, representations of numbers are spatially structured; comparing two numbers requires a cognitive 'zooming in' of a particular part of the axis, which is easier for numbers that are further apart. The size effect indicates that, unlike the number line taught at school, the mental number line is compressed logarithmically: as the numerical magnitudes increase, the distances between them decrease; thus, comparing smaller numbers is easier due to their greater resolution (see Cipora & Nęcka, 2012). Some studies suggest that during the course of individual development – specifically with the acquisition of school mathematical competence – the degree of logarithmic compression of the number line decreases, which is associated with a reduction in the strength of the distance and size effects. However, most researchers today agree that the cognitive representation of magnitude, including numerical magnitude, can be activated independently of space, which means that the distance and size effects cannot be used to confirm the existence of the mental number line.

Nevertheless, the theoretical construct of the mental number line has gained substantial support from the SNARC effect, first observed by Stanislas Dehaene and colleagues in 1993 and replicated several hundred times since (Dehaene, 2011; see also Cipora & Nęcka, 2012; Brożek & Hohol, 2017). The SNARC (Spatial-Numerical Association of Response Codes) effect, i.e., the spatial relationship between a number and a response type, is visible in experiments in which the respondent's task involves making decisions about the properties of numbers presented one after another (e.g., indicating whether a number is even or odd): shorter reaction times are observed for the left hand for relatively small numbers, and for the right hand for relatively large numbers. The numbers are relatively small or large because this effect can be observed even when the set of test stimuli includes numbers from 1 to 9 (the number 5 being omitted for methodological reasons). This pattern suggests that, as in the case of the number line taught in school, the mental representation of numbers is structured in a directional way: smaller numbers being more to the left and larger numbers to the right. Currently, debate is ongoing regarding which factors affect the strength of the SNARC effect and even its direction (see Butterworth, 2022). On the one hand, the role of cultural factors in the direction of the mental number line is emphasised. The standard SNARC effect is observed in cultures where writing and reading runs from left to right, while the reverse SNARC effect is observed in cultures where writing and reading runs from right to left. Moreover, when people are asked to count from 1 to 10 on their fingers, those who start with the left hand manifest a stronger effect than those who start with the right hand. On the other hand, the results of studies conducted on animals indicate that the association between numbers and space has deep evolutionary roots.

The distance and size effects indicate that numbers are cognitively represented in an analogue way, i.e., like physical quantities. The

SNARC effect is usually interpreted as an argument for linking mental representations of numbers and space. These conclusions are supported not only by the results of numerous behavioural studies but also by the results of neuroscience studies conducted using functional magnetic resonance imaging. For example, the intraparietal sulcus of the brain responds in a similar way when people process symbolic numbers and segments. Based on these and other results, Vincent Walsh has formulated a theory of magnitude called ATOM (A Theory of Magnitude), which assumes that in the parietal lobe cortex there is a common area of the metric mechanism which is involved in processing time, space, and numbers (see Butterworth, 2022). It is worth remembering that this does not mean that all these quantities are processed in exactly the same way or that their mechanisms do not involve additional components. Interestingly, the evidence supporting the role played by the parietal lobe in the processing of numbers and space emerged long before neuroimaging techniques were used. In the first half of the 20th century, the Austrian neurologist Josef Gerstmann described a set of symptoms (called Gerstmann syndrome) associated with damage to the angular gyrus. These include difficulties in processing numbers, distinguishing sides, and writing, as well as finger agnosia (e.g., impaired recognition of which of the patient's fingers is currently being stimulated).

Contemporary research aims to understand the mechanism of number processing not only at the level of brain structures but also the level of individual neurons. Studying macaques, Andres Nieder (2019) found numerical neurons located in the intraparietal sulcus area, i.e., an area specialised in size processing, and in the lateral prefrontal cortex, i.e., a structure involved in general cognitive processes. The name of these neurons is associated with the fact that they respond selectively to small non-symbolic numbers: a given neuron responds most strongly when a macaque perceives one dot; another neuron responds when the animal perceives two dots, and so on, up to four. Four (plus or minus one element) is a barrier to what is known as subitising, which is the effortless, rapid, and precise assessment of the number of elements exhibited. Subitising must not be confused with estimation, which involves approximating the number of elements in a set greater than four (plus or minus one) nor with counting. While subitising and estimation are independent of symbolic representations such as Arabic numbers or numerals, when we have learned counting, we are able to precisely determine the number of elements in a set that exceeds – and even significantly exceeds – the range of subitising. Although many studies have revealed that representatives of some animal species can be taught to count, researchers agree that this ability is part of man's cognitive ecology (see Butterworth, 2022).

In the context of the considerations presented in the *Historical analysis of the term* section, two questions arise: given current cognitive knowledge, can the elementary numerical abilities of subitising and estimation be considered the foundation on which counting, arithmetic, and other mathematical abilities are superstructured during ontogeny? Are these elementary abilities innate? In other words: were the philosophers who searched for the innate basis of mathematical intuition right, or was Hermann von Helmholtz, who questioned the existence of anything innate apart from the learning mechanisms themselves, right? Although debate concerning these questions is still ongoing (see Butterworth, 2022; Haman & Gut, 2016; Nieder, 2019), many researchers of mathematical cognition are inclined to support the thesis of the innateness of elementary numerical cognition and the thesis that it forms the basis for learning mathematics.

Regarding the innateness of elementary numerical competence, human infants manifest the ability to distinguish sets of objects that differ in number without the need for any training. This has been demonstrated by studies based on the habituation paradigm. When children aged about six months observe a series of pictures with an equal number of black dots that differ in size or placement, the period of their interest in the stimulus is shortened, which indicates habituation. When they are shown pictures with different numbers of dots, regardless of their spatial arrangement, they regain interest in the picture, as manifested by an extended period of attention to the stimulus (see Brożek & Hohol, 2017). Other studies have shown that infants possess an intuition of equinumerosity. For example, when hearing a sequence of four syllables, infants looked longer at a set of four dots than at a set of twelve dots. Furthermore, studies conducted on various animal species – ranging from non-human primates to rodents, birds, and even fish – indicate that the ability to perceive numerosity is common in nature, at least among vertebrates (see Butterworth, 2022; Nieder, 2019).

According to the influential and still developing theory of core knowledge formulated by Elizabeth Spelke (2022), our cognitive abilities are based on core knowledge systems. According to Spelke (2022), these systems

> operate on a limited domain of entities, and they capture only a limited subset of the properties that our perceptual systems deliver. These systems are also encapsulated, automatically activated and unconscious (…). These core knowledge systems emerge early in ontogeny and function throughout life, and they are present in people of all ages and in all cultures. Both during and after infancy, they guide our thinking and learning. Moreover, the same abilities and limits are found in a wide range of animals and depend on homologous brain systems and processes across those animals, providing evidence that core systems emerged deep in our evolutionary past (p. 190; see also Haman & Gut, 2016).

According to core systems theorists, two distinct systems located in the parietal lobe (called the object tracking system and the approximate number system), are responsible for subitising and estimation. Together, they serve as our 'number sense', which is a prerequisite for mastering numbers expressed symbolically and performing actions on them (see Dehaene, 2011).

As indicated above, there are arguments that support the thesis that representations of numbers are spatial in nature. But where do our spatial intuitions come from and are they, as Kant claimed, the basis of Euclidean geometry? According to Spelke (2022), as is the case with the 'number sense', our geometric abilities are grounded in two phylogenetically old, ontogenetically early, and culturally universal core systems, which are called the system of layout geometry and the system of object geometry (Hohol, 2020). The former is located in the hippocampus area and processes distances and directions, enabling navigation in the environment. The latter is located in the lateral structures of the occipital lobe and processes angles and lengths, enabling the recognition of two-dimensional shapes and three-dimensional objects. As Spelke (2022) observes, all core systems have certain limitations. The object tracking system provides accurate knowledge on numbers, but only up to about four objects. The approximate number system allows operations on larger numbers to be performed, but at the expense of precision. The system of layout geometry is sensitive to distance and direction, but not to angles; the system of object geometry processes lengths and angles, but not directions. How is it possible, then, that the cognitive representations generated by such limited systems are combined into a constraint-free system of mathematical concepts?

Although many details are still unknown, according to Spelke (2022), combining accurate representations of small numbers with approximate representations of larger numbers is possible thanks to the mastery of numerals. Children begin to use basic numerals at a very young age but cannot assign proper meanings to them until the age of two. A significant change occurs in the third year of life, when children begin to grasp that the numeral ‘one’ represents the abstract value of 1. This process continues for subsequent numbers and their corresponding numerals up to the number 4 in the fourth year of life. From this point onwards, the ability to use numerals develops rapidly, which goes hand in hand with learning how to solve more complex arithmetic tasks. Children formulate a universal rule that allows them to assign a corresponding numeral to each number, which promotes their understanding of both the cardinal and ordinal aspects of numbers. Language thus functions as a scaffolding to overcome the limitations of core number systems. Research further indicates that finger counting habits play an important role in the transition between knowledge contained in core systems and symbolic mathematics. Fingers are concrete objects that are easy to manipulate and are suitable for representing the numerosity of different objects. Moreover, due to their natural structuring, finger counting habits promote mastery of the ordinal aspect of numbers, whereas core knowledge systems only encode the cardinal aspect (see Szczygieł et al., 2015).

Research indicates that, in geometry, spatial vocabulary learning plays a similar role to numerals (see Hohol, 2020). This process is also supported by children’s growing experience with map-like sketches, through which children learn to represent the system of layout geometry, originally represented by the system of object geometry (direction, distance), as two-dimensional structures whose key geometric properties are angles and lengths.

Systematic reflection with conclusions and recommendations

Contemporary research on mathematical cognition can be seen as a continuation of philosophical inquiries into the nature of mathematical intuition. This is evidenced by the fact that Stanislas Dehaene and Elizabeth Brannon (2010) do not hesitate to call one of the main currents of research a 'Kantian research programme', which emphasises the continuity between the biopsychological foundations of mathematical cognition and how our minds process time and space, where Kantian 'filters' are replaced by core cognitive systems with a long evolutionary past. Dehaene and Brannon do not even hesitate to claim that if Kant were alive today, he would have become a neurocognitive scientist rather than a philosopher (p. 519). Compared to earlier philosophical inquiries, contemporary research on mathematical cognition makes it possible to go beyond the realm of rational speculation and to build theories that can be tested in experiments using methods that have also been proven to work well in the study of other areas of the mind. However, it would be rash to conclude that the empirical study on mathematical cognition simply replaces the philosophy of mathematics. While it is hardly conceivable today to build plausible philosophical theories about mathematical knowledge that would openly contradict the current understanding of neurocognitive mechanisms, it is equally hard to imagine neurocognitivists solving the problem of the ontological status of mathematical objects and structures. In other words, contemporary research on mathematical cognition cannot resolve the aforementioned dispute between Speusipppus and Menaechmus.

Another important strand is that empirical research on mathematical cognition sheds new light on a number of problems that basically all social sciences face. A key one relates to the question of 'nature or culture'. On the one hand, the foundations of mathematical cognition are extra-linguistic and directly related to the numerosity processing abilities and spatial skills possessed by many animals, which leads many researchers to consider them innate (see Butterworth, 2022; Haman & Gut, 2016). On the other hand, the innate form of mathematical cognition, i.e., core knowledge, is highly limited (e.g., the issue of the narrow scope of subitising). Transcending the limitations of core knowledge is possible thanks to a specifically human cultural product, i.e., language, as evidenced by the fact that speakers of languages characterised by narrow numeral systems, e.g., the Munduruku people, perform less precise calculations outside the scope of subitation (see Brożek & Hohol, 2017; Butterworth, 2022). The classic 'nature or culture' question thus seems misplaced to the extent that, without innate core systems, it could be argued that we would acquire no mathematical practices through education at all. However, these practices are deeply immersed in culture, the products of which act as 'cognitive tools' that significantly modify core knowledge.

There are also a number of important practical implications for education associated with the study of mathematical cognition. First and foremost, experts in the field of mathematical cognition recommend introducing mathematical abstractions at school by grounding them in spatial experience, which is common to all humans due to the structure of our bodies. Particularly important in this respect is the promotion of the formation of spatial-numerical associations. However, the thesis 'more space, better mathematics' is not universally true. While stronger spatial-numerical associations in children (expressed, e.g., by the aforementioned stronger SNARC effect) are usually correlated with better competence in school mathematics, the relationship may be reversed in adulthood. This is particularly evident in the case of professional mathematicians, who do not manifest the SNARC effect at all, which may indicate that prolonged training leads to the construction of more abstract, or less embodied, representations of numbers (Cipora et al., 2016).

Moreover, research on 'number sense' is valuable for understanding the nature of developmental dyscalculia, which hinders and, in extreme cases, even prevents the acquisition of school mathematical knowledge and is a barrier to entry into numerous vocational pathways. As cyclically published educational reports indicate, learning difficulties in mathematics have far-reaching consequences for the well-being of entire societies (see Butterworth, 2022, pp. 30–33). Thus, psychological interventions are recommended as early as possible to help those with diagnosed dyscalculia or at risk of dyscalculia in order to stimulate the fullest possible development of their number sense, which is a prerequisite for the acquisition of school mathematical knowledge (see Dehaene, 2011; Butterworth, 2022).

Finally, research in the field of mathematical cognition demonstrates that the processing of mathematical knowledge, which is considered the most abstract object of human cognition, is, at its core, based on the mechanisms used in everyday life by everyone, regardless of their educational and professional paths.

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